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NUMERICAL GREEN'S FUNCTION APPROACH TO FINITE-SIZED PLATE ANALYSIS

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Abstract—The eigenfunction expansion method is used to obtain numerical Green's functions to solve for deflection of irregular-shaped classical plates. The associated eigenvalue problem allows to express the Green's function as a series of eigenfunctions which are approximated by a series of polynomials that satisfy the homogeneous boundary conditions to which the plates are subjected. A computer algebra system (Mathematica) has been extensively used to construct the approximate Green's functions consisting of polynomials, reducing substantially the amount of work involved in the calculation and achievement of the solution. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Classical plate theory has been investigated for a long time since the last century and numerous analytical techniques have been developed exemplified by Levy's method and Navier's method (see Reddy, 1984). However, the shape and the boundary conditions for which those analytical methods can be used are very limited and purely numerical methods such as the finite element method are routinely used.

The Green's function approach has been well-known in micromechanics but little is known about the Green's function for plates except for simple cases such as homogeneous and infinitely-extended plates (applied to the boundary element method).

Melnikov (1977) used a Green's function approach for a body whose boundary shape does not coincide with the coordinate surface based on the Green's function available for regular shapes. Irschik and Ziegler (1981) applied the Green's function method for a polygonal plate by embedding in a rectangular domain by applying coincidence of boundaries as possible. Qin *et al.* (1991) investigated a classical plate that was extended to infinity with an inclusion at the center and concluded that the stress field inside the inclusion becomes uniform. Nomura and Choi (1994) showed a Green's function approach for 2-D elasticity problems as the base for the present approach.

The advantage of the Green's function approach is that once the Green's function is found, the deflection can be expressed by the convolution type integral between the Green's function and distributed lateral loads. The Green's function solution contains all the boundary conditions so the solution can be evaluated without extra calculations when the boundary conditions are altered.

In this paper, a systematic way of constructing Green's functions for Kirchhoff type plates with irregular shapes is presented. This paper is the first attempt to derive numerical Green's functions for such plates to the authors' best knowledge. The Green's function for plates can be expressed by eigenfunctions for the associated eigenvalue problem. The eigenfunctions are then expressed by a linear combination of trial functions each of which satisfies the homogeneous boundary conditions. The Galerkin's method is employed to obtain the coefficients of the trial functions.

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Determination of trial functions that may satisfy the homogeneous boundary conditions and manipulation of the trial functions in the Galerkin procedure call for a tremendous amount of algebra for which the use of a computer algebra system, i.e. Mathematica (1988) is essential.

It is found that numerical results for the deflection of classical plates give almost identical values to the available classical methods for regular geometry (rectangles). As a demonstration of the present approach, deflection for triangular plates is sought using the Green's function approach.

2. SOLUTION PROCEDURE

From the classical plate theory the general differential equation for plate deflection is given by:

$$Lw(x,y) = q(x,y), \tag{1}$$

where $w(x,y)$ is the deflection of the plate, $q(x,y)$ is a distributed load function and L is the biharmonic operator defined as

$$L = D\Delta\Delta, \tag{2}$$

where D is the flexural rigidity of the plate and Δ is the (2-D) Laplacian operator.

Under the condition that there are no edge preloads, the unknown $w(x,y)$ is expressed by

$$w(x,y) = \iint_R G(\xi,\eta\,;x,y)q(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{3}$$

where R is over the plate surface. The function, $G(\xi,\eta\,;x,y)$, is the Green's function which satisfies

$$L^*G(\xi,\eta\,;x,y) = \delta(x-\xi,y-\eta), \tag{4}$$

where $\delta(x-\xi,y-\eta)$ is the two-dimensional Dirac delta function and L^* is the adjoint operator of L but in this case $L = L^*$ (self-adjoint). The Green's function, $G(\xi,\eta\,;x,y)$, must satisfy the homogeneous boundary condition.

It can be shown that the Green's function, $G(\xi,\eta\,;x,y)$, is expressed by

$$G(\xi,\eta\,;x,y) = \sum_{k=1}^{\infty}\frac{\phi_k(x,y)\phi_k(\xi,\eta)}{\lambda_k}, \tag{5}$$

where $\phi_k(x,y)$ and λ_k are a k-th eigenfunction and eigenvalue, respectively, for the following eigenvalue problem.

$$L\phi(\xi,\eta) = \lambda\phi(\xi,\eta). \tag{6}$$

It should be noted that because of the self-adjoint nature of L, each eigenvalue is positively defined and

$$\iint_R \phi_i(\xi,\eta)\phi_j(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = \delta_{ij}, \tag{7}$$

where δ_{ij} is the Kronecker delta.

The Galerkin method is used for the calculation of the eigenfunctions and the eigenvalues so that the eigenfunctions can be approximated in terms of polynomials. The eigenfunctions may be expressed as a linear combination of trial function as

$$\phi_k(x,y) = \sum_{j=1}^{n} c_{kj} f_j(x,y), \tag{8}$$

where c_{kj} are the undetermined coefficients and the trial functions, $f_j(x,y)$ (polynomials in x and y), satisfy the prescribed homogeneous boundary conditions. Their form is given by

$$f_j(x,y) = F_{00}+F_{10}x+F_{01}y+F_{20}x^2+F_{11}xy+F_{02}y^2+\cdots, \tag{9}$$

where F_{ji} are unknown coefficients to be determined. Substituting the eigenfunctions expressed by eqn (8) into (6), multiplying both terms by f_i and integrating over the entire area yield

$$\sum_{j=1}^{n} A_{ij}c_{kj} = \lambda_k \sum_{j=1}^{n} B_{ij}c_{kj}, \tag{10}$$

or

$$A\mathbf{c}_{(k)} = \lambda_k \mathbf{B}\mathbf{c}_{(k)}, \tag{11}$$

where

$$A_{ij} = \int\int f_i L f_j \,\mathrm{d}x\,\mathrm{d}y, \quad B_{ij} = \int\int f_i f_j \,\mathrm{d}x\,\mathrm{d}y, \tag{12}$$

and $\mathbf{c}_{(k)}$ is the vector,

$$\mathbf{c}_{(k)} = \begin{pmatrix} c_{k1} \\ \vdots \\ c_{kn} \end{pmatrix}.$$

Equation (11) is a generalized eigenvalue problem whose solution technique is readily available. The matrix **B** is symmetrical and positive definite. In general, for very large matrices the problem may be solved most efficiently by using the Cholesky decomposition.

3. NUMERICAL EXAMPLES

This section presents some numerical examples of triangular plates with clamped and simply-supported edges in order to demonstrate the developed procedure.

It is necessary to choose the trial functions for the given geometry and boundary conditions. If the region is expressed by a set of curved lines ($f_1(x,y) = 0, f_2(x,y) = 0\ldots$), the trial function of eqn (9) for simply supported and clamped edges can be expressed as $f_1(x,y)\times f_2(x,y)\times\ldots(F_{00}+F_{10}x+F_{01}y+F_{20}x^2+F_{11}xy+F_{02}y^2+\ldots)$. The unknown coefficients (F_{ij}) can be determined so as to satisfy the remaining boundary conditions ($\partial^2 w/\partial^2 n = 0$ for simply supported edges or $\partial w/\partial n = 0$ for clamped edges). For free edges, $f_j(x,y)$'s are assumed to be polynomials in the form of $F_{00}+F_{10}x+F_{01}y+F_{20}x^2+F_{11}xy+F_{02}y^2+\ldots$ and each coefficient is chosen to satisfy the free-edge boundary conditions. In either case, a set of underdetermined simultaneous equations need to be solved for the unknowns F_{ij} that can be performed symbolically or numerically on a routine basis. The minimum order of polynomials can be determined so that the underdetermined equations have at least one set of solution. A computer algebra system can be used to expand $f_i L f_j$ in the form of $\Sigma\Sigma b_{ij}x^i y^j$ so that integration of $f_i L f_j$ over

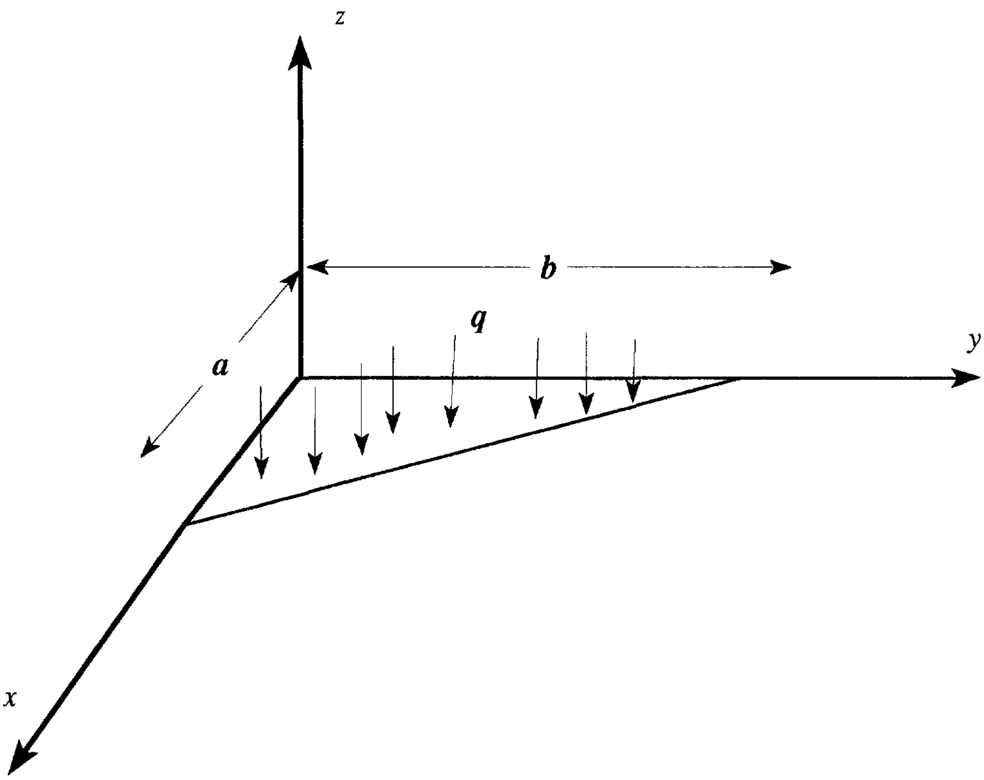

Fig. 1. Triangular plate under uniformly distributed load.

the given region is reduced to integration of $x^i y^j$ which can be carried out analytically for a wide variety of shapes.

As a first test of the present method, the Navier's solution for deflection of classical plates (Kirchoff type) with simply supported edges was compared with the Green's function method. As far as numerical results are concerned, the present approach yielded numerically identical results with the Navier's method which serves as verification of the present method.

Next, the analysis was performed for an isosceles right-angle (90°) triangle with two equal sides, therefore $a = b = 1$ as shown in Fig. 1. Poisson's ratio values (v) of 3/10 and 1/6 are used to match the results found in the literature ([1, 4]). The load q is uniformly distributed. The boundary conditions for the triangular plate shown in Fig. 1 are as follows:

—for a clamped edge

$$w = 0, \quad \frac{\partial w}{\partial n} = 0, \tag{13}$$

—for a simply supported edge

$$w = 0, \quad \frac{\partial^2 w}{\partial n^2} = 0, \tag{14}$$

where n is the normal to the boundary.

3.1. *Triangular plate with all three edges clamped*

Tables 1–6 are comparisons between the present method and published values where N is the number of polynomial terms used. Table 1 shows the values for deflection at $(x, y) = (a/3, b/3)$ for different polynomial orders for the triangular plate with all the edges clamped.

Table 1. Comparison of $w(1/3, 1/3)$ and difference % for a triangular plate with all three edges clamped

Degree	Number of polyn.	$-wD/qa^4$ at $x = a/3$, $y = b/3$	Comparison with Lekhnitskii (%)
6	1	0.00014289	0.07
7	3	0.00016633	—
8	6	0.00017652	—
9	10	0.00017670	—
10	15	0.00017784	—
11	21	0.00017785	—

Table 2. Maximum moments for a triangular plate with all three edges clamped ($N = 15$ and $N = 21$)

	M_x/qa^2 at $x = a/3$, $y = b/3$	Comparison with Bares (%)	M_y/qa^2 at $x = a/3$, $y = b/3$	Comparison with Bares (%)
degree = 10	0.0080354	−0.043	0.0080354	−0.0043
degree = 11	0.0080039	−0.047	0.0080039	−0.047

Table 3. Shear forces for a triangular plate with all three edges clamped ($N = 15$ and $N = 21$)

	V_x/qa max	Comparison with Bares (%)	V_y/qa max	Comparison with Bares (%)
degree = 10	−0.231339	−17.9	−0.231339	−17.9
degree = 11	−0.229854	−18.4	−0.229854	−18.4

Table 4. Comparison of $w(1/3, 1/3)$ and difference % for a triangular plate with all three edges simply supported

Degree	Number of polyn.	$-wD/qa^4$ at $x = a/3$, $y = b/3$	Comparison with Mansfield (%)
6	1	0.00007019	—
7	2	0.00020785	—
8	5	0.00049241	−23.3
9	9	0.00063198	−1.62
10	14	0.00063871	−0.57
11	20	0.00064125	−0.18

Table 5. Maximum moments for a triangular plate with all three edges simply supported ($N = 14$ and $N = 20$)

	M_x/qa^2 at $x = a/3$, $y = b/3$	Comparison with Bares (%)	M_y/qa^2 at $x = a/3$, $y = b/3$	Comparison with Bares (%)
degree = 10	0.0186285	−1.43	0.0018747	−0.08
degree = 11	0.0187156	−0.09	0.0186106	−1.53

Table 6. Shear forces for a triangular plate with all three edges simply supported ($N = 14$ and $N = 20$)

	V_x/qa max	Comparison with Bares (%)	V_y/qa max	Comparison with Bares (%)
degree = 10	−0.194751	−27.1	−0.187014	−29.9
degree = 11	−0.295335	10.61	−0.286648	7.36

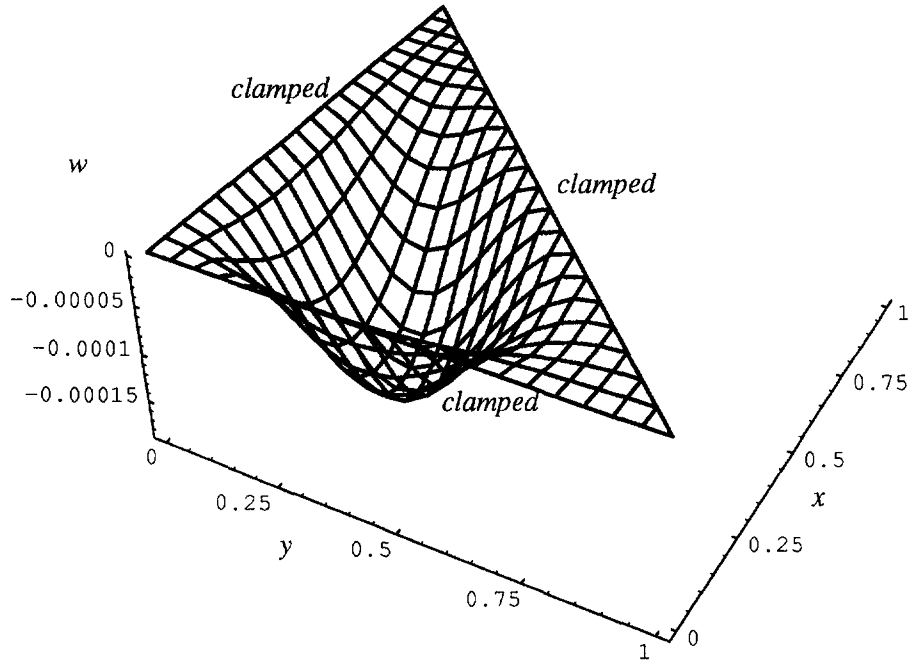

Fig. 2. Deflection for a triangular plate with three edges clamped ($N = 15$).

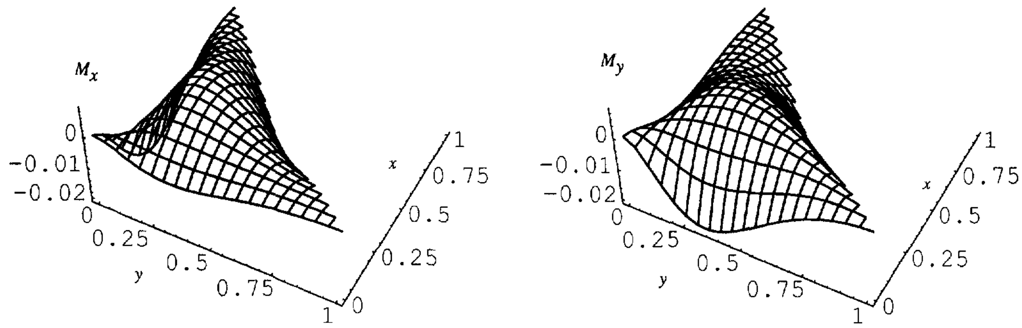

Fig. 3. Moments in x and y for a triangular plate with all three edges clamped ($N = 15$).

The only numerical value available is $w = -0.000143qa^4/D$ reported by Lekhnitskii (1987) using one polynomial approximation by the Rayleigh-Ritz method which should be favorably compared with $w = -0.00014289qa^4/D$ in the present approach for the lowest order approximation. As is seen from the table, the deflection values seem to converge around $w = -0.0001778qa^4/D$.

Tables 2 and 3 show, respectively, the values for the maximum moment and shear force. The values for the maximum bending moment obtained by Bares (1971) were $M_x = M_y = 0.00840qa^2$ for $v = 1/6$. Bares also obtained values for the maximum shear as $V_x = V_y = -0.282qa$ for $v = 1/6$.

Figure 2 shows the deflection distribution over a triangular plate with all three edges clamped, while Figs 3 and 4 show, respectively, the bending moment, and the shear stress distribution in the x and y directions for the same type of plates.

3.2. *Triangular plate with all three edges simply supported*

In this example, a triangular plate is simply supported at all the edges. Table 4 shows the numerical value of deflection of the triangular plate for different orders of polynomials.

Lekhnitskii (1987) obtained $w = -0.000619qa^4/D$ using three polynomials. Numerical values using double trigonometric series expression given by Mansfield (1989) were $w = -0.00064273$, -0.000642494, and $-0.000642479qa^4/D$ for $(m, n) = (7, 8)$, $(9, 10)$, and

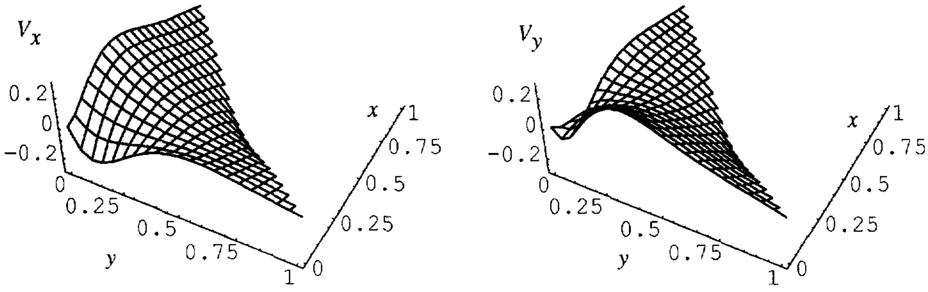

Fig. 4. Shear forces in x and y for a triangular plate with all three edges clamped ($N = 15$).

(11, 12), respectively. Bares (1971) obtained $w = -0.000667932qa^4/D$ for $v = 1/6$. The method by Timoshenko and Woinowsky-Krieger (1959) yielded $w = -0.000642508qa^4/D$ independent of v which is favorably compared with the present result.

The values for the maximum moments and shear force are shown in Tables 5 and 6, respectively. The maximum bending moment values obtained by Bares (1971) were $M_x = 0.00189qa^2/D$ and $M_y = 0.00189qa^2/D$ for $v = 1/6$. Bares (1971) obtained maximum shear values of $V_x = -0.267qa$ and $V_y = -0.267qa$ for $v = 1/6$.

4. CONCLUSIONS

The associated eigenvalue problem leads to expressing the Green's function as a series of eigenfunctions which are approximated by a series of polynomials satisfying the homogeneous boundary conditions. The Galerkin method was preferred over other weighted residual methods to obtain the eigenfunctions because it ensures the convergence of the solution. Although it was not included in this paper, few runs were made using the least square method showing a slower convergence. The undetermined coefficients of the approximate eigenfunctions are the obtained eigenvectors.

With the Green's function, it is possible to express the deflection as a convolution type integral between the Green's function and the distributed load ; thus redundant computation can be avoided when the lateral load is altered. In addition, preloads at the edges of the plate can be easily incorporated into a boundary integral between the derivative of the Green's function and the preload.

The use of polynomials as trial functions in this paper constitutes another deviation from the conventional approaches found in the literature. Most of the computation was performed using a computer algebra system, Mathematica, which can automate the normally tedious and time-consuming algebra involved in the present paper.

It stands to reason that cases of plates with a greater number of edges (sides) as well as skewed plates may also be worked using the eigenfunction expansion-based Green's function method. The most time-consuming part of the present approach is the generation of eigenfunctions that need to be performed symbolically. Once such a routine is written by a computer algebra program, the rest of the numerical computation can be carried out by conventional compilers that can be optimized. The method has been limited in this paper to straight edges although it may be modified for curved edges. Diaz-Contreras (1994) have also adapted the approach presented in this paper to solve for non-conventional plate theories.

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